Octahedral Graph Scaling

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Abstract

There is presently no strong interpretation for the notion of n-vertex graph scaling. This paper presents a new definition for the term in the context of "Octahedral Scaling" or 4-vertex graph scaling, that may be considered for other values of n, and presents two different calculation methods based on two different interpretations of scale; one from fractal geometry and the other from the geometry of platonic solids. A conjecture is presented as to the uniqueness of the Pythagorean series to form natural graphs and its relevance to mathematical cosmology.

1 Introduction

n-vertex graphs such as the 3-vertex cubical and dodecahedral graphs, and the 4-vertex octahedral graph have no inherent scaling property and so a method for determining a scaling factor for them seems not to exist in current literature. Scaling factors are found in fractal geometry, and a number of different methods exist to determine scale given different fractal geometries.

My original goal was to find a suitable series on the counting numbers that could be formed into a graph where its scale could be determined. This is a component step in the eventual construction of a causal set based on the series.

Even though there are many "graph" like structure in various branches or mathematics they are in the main entirely man-made and artificial representations that flow from artificial axiomatic spaces. I was looking for a "natural" graph that had its roots close to some innate natural aspect of the counting numbers. In the analysis of the Pythagorean series of unique hypotenuse based on

$$a^2 + b^2 = c^2$$

the Simple Tree of Pythagorean Triples seems to form a graph suitable for study. Locally each node in the graph is 4-vertex, and the scale of the graph could be determined by examining the ratio of the densities of members between generations of the graph.

A stepwise numerical calculation in a spread sheet based on the three matrix Barning method [2] showed that the ratio appears to asymptote to 0.5147186257 with each generation of the graph. I then searched using Google for this number to see if it had any significance and found "Inside a cube - solutions" [1] which presents the standard result to the question of platonic solids inside the unit cube.

This fractal scaling method derived decimal is very close to

$$3(3-2\sqrt{2})$$

which represents the diameter of each of six spheres (octahedral close packed), inside the unit cube. Given that I was dealing with a 4-vertex graph in the Simple Tree of Pythagorean Triples, a result that indicated a relationship with 4-vertex platonic solid scaling seemed interesting.

If it could be proven that the scale of the tree of primitive Pythagorean triples was equal to $3(3-2\sqrt{2})$, by solving the infinite matrix sequence, then this would indicate that the notion of scale from the unit cube packing example for an octahedron, and the notion of scale from the tree were mutually reinforcing.

I posted this question on StackExchange Mathematics and was delighted by a succinct proof.

2 Tree of Primitive Pythagorean Triples

The question "Tree of Primitive Pythagorean Triples graph scale infinite series" was posted by me on StackExchange Mathematics[3].
The proof is provided by a solution from Ricardo Buring (as Community Wiki).

2.1 Question

Consider the tree of primitive Pythagorean triples as seen here:

https://en.wikipedia.org/wiki/Tree_of_primitive_Pythagorean_triples

Consider the values for c in the triples (a, b, c) and their density in each generation layer of the tree. Taking the number of c's in any generation and dividing by the difference between the maximum and minimum values for c in that generation gives the density of that generation.

The "scale" of the graph is given by the ratio of densities as the generations go on to infinity. I have calculated the maximum, minimum and number of c's in increasing generations of the tree and the 'scale' asymptotes to 0.5147186257? by calculation.

Note: This is a graph scaling question and is not the same question as the simple density of primitive triples as solved by Lehmer (1900).

I postulate that this graph scaling value is in fact $3(3-2\sqrt{2})$. Can this be proved?

2.2 Answer

If (a, b, c) is a primitive Pythagorean triple, then the c values of its descendants (by the matrix expression) are

$$2a - 2b + 3c, \quad 2a + 2b + 3c, \quad -2a + 2b + 3c$$

Clearly the middle one is the largest. Hence the middle one is the largest in any generation. With the starting value $(3, 4, 5)$, the first is the smallest. Hence the first is the smallest in any generation.

In the notation of Wikipedia and by diagonalization we have

$$A^n = \begin{pmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 2n^2+1 \end{pmatrix}$$

Hence the minimum c value in the nth generation is

$$c_n^{min} = 2n^2 + 6n + 5$$

Similarly but more horrifically (diagonalizing B), the maximum c value in the nth generation is

$$c_n^{max} = \frac{1}{2}\left(5 - \frac{7}{\sqrt{2}}\right)(3 - 2\sqrt{2})^n + \frac{1}{2}\left(5 + \frac{7}{\sqrt{2}}\right)(3 + 2\sqrt{2})^n$$

The density of the nth generation is

$$d_n = \frac{3^n}{c_n^{max} - c_n^{min}}$$

The quotient of two consecutive densities is

$$q_n = \frac{d_{n+1}}{d_n} = 3\frac{c_n^{max} - c_n^{min}}{c_{n+1}^{max} - c_{n+1}^{min}}$$

Finally, we have

$$q = \lim_{n\to\infty} q_n = 3\lim_{n\to\infty} \frac{c_n^{max} - c_n^{min}}{c_{n+1}^{max} - c_{n+1}^{min}} = 3(3 - 2\sqrt{2})$$

The last limit was computed using a CAS

3 Octahedral Explosion of the Unit Sphere in the Unit Cube

The optimal packing of spheres in the unit cube is given by considering the positioning of the three orthogonal squares that are at the heart of the octahedron. The diameters of each of the resulting six spheres is $3(3 - 2\sqrt{2})$.

This is a standard geometric result [1].

The ratio of the ending diameter to the starting diameter (1 in the unit cube) gives the scale factor. In this case

$$s = \frac{\varnothing_{6spheres}}{\varnothing_{1sphere}} = 3(3 - 2\sqrt{2})$$

It is a convenient notion that platonic solids scale by virtue of their explosion in the unit cube and that this scale is given by the ending diameters of the spheres so packed at the vertices.

4 Conclusion

A relationship between the primitive Tree of Pythagorean Triples and the octahedral graph has been established. These two graphs share the following properties:

1. 4-vertex
2. scale factor $s = 3(3 - 2\sqrt{2})$

The Simple Tree of Pythagorean Triples is a form of open graph and the octahedral graph is a form of closed graph. The scale factor of the Simple Tree Pythagorean of Triples was calculated using a fractal scale determination method via the limit of the infinite series of ratios of the density of members in each successive generation of the graph.

The scale factor of the octahedral graph was calculated using the sphere explosion in unit cube method (being postulated in this paper) via traditional 3D geometry.

The scales where found to be identical.

5 Further Work

The meaning of the result to future researchers is that traditional Euclidian results can also be obtained by considering solids in their alternative graph (open or closed) formulations where the limiting values of some graph based features can be shown to be representative of the properties of 3D projected solids.

It is of interest that there is another underlying common property of both calculation methods. The fractal scaling method involves a projection to infinity by the taking of a limit in order to establish the value of scale. The unit cube method also involves taking a projection from a two dimensional octahedral graph to a three dimensional octahedron solid in order to recover the concept of scale.

Scale is not apparent as a property of either the open (tree) or closed (graph) until a form of dimensional (+1) projection is taken.

The result was obtained exclusively for the 4-vertex graph in the octahedral context only. Further study is needed to extend the result where possible to other n-vertex graphs and their associated 3 space solid projections.

Due to this result the properties of octahedral graphs and octahedrons may be considered more viable as the basis of natural causal set construction and dynamics. In consideration of the varied candidate series on the counting numbers, I would postulate that a filter based upon the ability of the series to form a fixed vertex count graph that has both an open and closed form and a calculable scale is an important discriminant. In this case the series is based on the tree of primitive Pythagorean triples and the graph is octahedral, both open and closed with calculable scale.

Further work is required on the nature and properties of natural series and their ability to form graphs to understand whether this example is the only one (the conjecture), or simply a member of a class of series that have this behaviour. Should this conjecture prove true then this would be an important result to mathematical cosmology as it illuminates a unique pathway from the counting numbers to higher order mathematical constructs.

References

[1] Philippe Chevanne. Inside a cube - solutions.

[2] Wikipedia Community. Tree of primitive pythagorean triples.

[3] Peter Russell. Tree of primitive pythagorean triples graph scale infinte series.